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ScienceDirect
J. Differential Equations 242 (2007) 11–23
Journal of Differential Equations
www.elsevier.com/locate/jde

Sufficient conditions for the existence of at least n or exactly n limit cycles for the Liénard differential systems

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Received 20 January 2007; revised 10 July 2007

Available online 6 September 2007

Abstract

In this paper we study the limit cycles of the Liénard differential system of the form $\ddot{x} + f(x)\dot{x} + g(x) = 0$, or its equivalent system $\dot{x} = y - F(x)$, $\dot{y} = -g(x)$. We provide sufficient conditions in order that the system exhibits at least n or exactly n limit cycles.

MSC: 58F21; 34C05; 58F14

Keywords: Liénard polynomial vector fields; Liénard systems; Limit cycles

1. Introduction and statement of the results

Hilbert [8] in 1900 and in the second part of its 16th problem proposed to find an estimation of the uniform upper bound for the number of limit cycles of all polynomial differential systems of a given degree, and also to study their distribution or configuration in the plane. Except for the Riemann hypothesis, the 16th problem seems to be the most elusive of Hilbert's problems. It has been one of the main problems in the qualitative theory of planar differential equations in

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[1] The second author is partially supported by a MCYT/FEDER grant No. MTM2005-06098-C02-01 and by a CICYT grant No. 2005SGR 00550.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.07.004

the XX century. The contributions of Écalle [7] and Ilyashenko [9] proving that any polynomial differential system has finitely many limit cycles have been the best results in this area. But until now it is not proved the existence of an uniform upper bound. This problem remains open even for the quadratic polynomial differential systems. However, it is not difficult to see that any configuration of limit cycles is realizable for some polynomial differential system, see for details [12].

Thus we have the finiteness of the number of limit cycles for every polynomial differential system of degree m, but we do not have uniform bounds for that number in the whole class of all polynomial differential systems of degree m. Following to Smale [15] we consider a more easy and special class of polynomial differential systems, the *polynomial Liénard systems*:

$$\dot{x} = y - F(x), \qquad \dot{y} = -x, \tag{1}$$

where $F(x) = a_{m-1}x + \cdots + a_0 x^m$. For these systems the existence of uniform bounds also remain unproved. But when the degree m of these systems is odd Ilyashenko and Panov in [10] obtained an uniform upper bound for the number of limit cycles in a subclass of systems such that F is monic and its coefficients satisfy some estimations.

For the Liénard systems (1) Lins, de Melo and Pugh [11] conjectured that they have at most k limit cycles if $F(x)$ is a polynomial of degree $m = 2k + 1$ or $m = 2k + 2$. This conjecture is supported mainly by the following three facts. First, the Liénard systems of the form

$$\dot{x} = y - \varepsilon F(x), \qquad \dot{y} = -x,$$

with ε sufficiently small have at most k limit cycles bifurcating from the periodic orbits of the linear center $\dot{x} = -y$, $\dot{y} = x$, and there are examples with exactly k, see [11]. Second, it is known that systems (1) have a center at the origin if and only if $a_i = 0$ for all i's odd, and that these a_i with i odd are the Liapunov constants of systems (1). Consequently at most k small limit cycles can bifurcate by Hopf from these centers, when we perturb them inside the class of all Liénard systems of degree $m = 2k + 1$ or $2k + 2$, see Zuppa [22], and also Blows and Lloyd [2]. Third, López and López-Ruiz [13] have studied the Liénard systems (1) in what they call the strongly nonlinear regime. In this regime they show that the conjecture is true when m is odd. More recently it was proved by Xiudong Chen and Yong Chen in [4] that the conjecture holds restricted to Liénard systems (1) with the function $F(x)$ odd.

In [6] the authors state that for a well-chosen polynomial $F(x)$ of degree 7, the Liénard system (1) exhibits 4 limit cycles, instead of the 3 conjectured by Lins, de Melo and Pugh.

In this paper we provide sufficient conditions in order that the Liénard system (1) exhibits at least n or exactly n limit cycles with $n \leqslant k$. But in our study $f(x)$ and $g(x)$ do not need to be polynomial. There are many results on the limit cycles of Liénard systems, see for instance [16,18,21].

The classical Liénard differential equation is

$$\ddot{x} + f(x)\dot{x} + x = 0. \tag{2}$$

It is equivalent to the differential system

$$\dot{x} = y - F(x), \qquad \dot{y} = -x, \tag{3}$$

taking

$$F(x)=\int_0^x f(s)\,ds.$$

We assume

(H1) $g(x), f(x)\in\mathcal{C}^0(\mathbb{R})$, and
(H2) $xg(x)>0$ when $x\neq 0$.

Let $x_1(z)$ be the inverse function of $z=G(x)$ in $x\geqslant 0$, where

$$G(x)=\frac{1}{2}x^2.$$

Similarly let $x_2(z)$ be the inverse function of $z=G(x)$ in $x\leqslant 0$. Then we define

$$F_1(z)=F\bigl(x_1(z)\bigr)\quad\text{and}\quad F_2(z)=F\bigl(x_2(z)\bigr).$$

Now the differential system (3) can be written as

$$\frac{dz}{dy}=F_1(z)-y,\quad \text{if } x\geqslant 0,\quad \text{and} \tag{4}$$

$$\frac{dz}{dy}=F_2(z)-y,\quad \text{if } x\leqslant 0. \tag{5}$$

Our first result provides sufficient conditions for the existence of at least one limit cycle for the Liénard system (3).

Theorem 1. *Consider the Liénard system* (3) *satisfying* (H1) *and* (H2), *or its equivalent systems* (4) *and* (5), *if for system* (4) *there exist* $z_{11}>z_{10}>0$ *and for systems* (5) *there exist* $z_{21}>z_{20}>0$ *satisfying*

$$M_{10}+\sqrt{2z_{10}}\leqslant\max\Bigl\{M_{20},m_{20}+\sqrt{\bigl(m_{20}-F_2(z_{20})\bigr)^2+2z_{20}}\Bigr\},$$

$$m_{20}-\sqrt{2z_{20}}\geqslant\min\Bigl\{m_{10},M_{10}+\sqrt{\bigl(M_{10}-F_1(z_{10})\bigr)^2+2z_{10}}\Bigr\},$$

$$M_{21}+\sqrt{2z_{21}}\leqslant\max\Bigl\{M_{11},m_{11}+\sqrt{\bigl(m_{11}-F_1(z_{11})\bigr)^2+2z_{11}}\Bigr\},$$

$$m_{11}-\sqrt{2z_{11}}\geqslant\min\Bigl\{m_{21},M_{21}+\sqrt{\bigl(M_{21}-F_2(z_{21})\bigr)^2+2z_{21}}\Bigr\},$$

then system (3) *has at least one limit cycle, where* $m_{ij}=\min_{z\in[0,z_{ij}]}F_i(z)$ *and* $M_{ij}=\max_{z\in[0,z_{ij}]}F_i(z)$ *for* $i,j=1,2$.

Theorem 1 is proved in Section 2.

Corollary 2. *Under the assumptions of Theorem* 1 *if for system* (4) *there exist* $z_{11} > z_{10} > 0$*, and for system* (5) *there exist* $z_{21} > z_{20} > 0$*, such that*

(i) $M_{10} + \sqrt{2z_{10}} \leqslant M_{20}$, $m_{20} - \sqrt{2z_{20}} \geqslant m_{10}$;
(ii) $M_{21} + \sqrt{2z_{21}} \leqslant M_{11}$, $m_{11} - \sqrt{2z_{11}} \geqslant m_{21}$;

then system (3) *has at least one limit cycle.*

In our main result, Theorem 3, we provide sufficient conditions to guarantee the existence of at least n, or exactly n limit cycles for the Liénard system (3). We remark that it is not necessary that $F(x)$ be an odd function as in the papers [1,14].

Theorem 3. *Consider the Liénard system* (3) *satisfying* (H1) *and* (H2)*, or its equivalent systems* (4) *and* (5)*. We assume the following four conditions.*

(i) *Each* $F_i(z)$ $(i = 1, 2)$ *has n positive zeros,* $F_1(x_j) = 0$ *and* $F_2(x_{-j}) = 0$ *for* $j = 1, \ldots, n$*. Moreover there exists* Δ_j *either in* (x_j, x_{-j})*, or* (x_{-j}, x_j) *such that* $F_1(\Delta_j) = F_2(\Delta_j)$ *for* $j = 1, \ldots, n$; *see Fig.* 2*.*
(ii) *Each* $F_i'(z)$ $(i = 1, 2)$ *also has n positive zeros* $F_1'(\alpha_j) = 0$ *and* $F_2'(\alpha_{-j}) = 0$ *for* $j = 1, \ldots, n$*. Moreover, if n is odd then*

$$0 < F_1(\alpha_1) < F_1(\alpha_3) < \cdots < F_1(\alpha_n),$$
$$0 > F_1(\alpha_2) > F_1(\alpha_4) > \cdots > F_1(\alpha_{n-1}),$$
$$0 > F_2(\alpha_{-1}) > F_2(\alpha_{-3}) > \cdots > F_2(\alpha_{-n}),$$
$$0 < F_2(\alpha_{-2}) < F_2(\alpha_{-4}) < \cdots < F_2(\alpha_{-n+1});$$

if n is even then

$$0 < F_1(\alpha_1) < F_1(\alpha_3) < \cdots < F_1(\alpha_{n-1}),$$
$$0 > F_1(\alpha_2) > F_1(\alpha_4) > \cdots > F_1(\alpha_n),$$
$$0 > F_2(\alpha_{-1}) > F_2(\alpha_{-3}) > \cdots > F_2(\alpha_{-n+1}),$$
$$0 < F_2(\alpha_{-2}) < F_2(\alpha_{-4}) < \cdots < F_2(\alpha_{-n}).$$

(iii) *For* $k = 0, 1, 2, \ldots, n-1$ *we have that*

$$F_2(\alpha_{-2k-2}) - F_1(\alpha_{2k+1}) \geqslant \sqrt{2\alpha_{2k+2}},$$
$$F_2(\alpha_{-2k-1}) - F_1(\alpha_{2k+2}) \geqslant \sqrt{2\alpha_{-2k-2}}.$$

Moreover for $k = 1, 2, 3, \ldots, n-1$ *we have that*

$$F_1(\alpha_{2k+1}) - F_2(\alpha_{-2k}) \geqslant \sqrt{2\alpha_{-2k-1}},$$
$$F_1(\alpha_{2k}) - F_2(\alpha_{-2k-1}) \geqslant \sqrt{2\alpha_{2k+1}}.$$

(iv) *There exist* $\beta_j \in (x_j, \alpha_{j+1})$ *and* $\beta_{-j} \in (x_{-j}, \alpha_{-j-1})$ *such that* $F_1(\beta_j) = F_1(\alpha_{j-1})$ *and* $F_2(\beta_{-j}) = F_2(\alpha_{-j+1})$ *for* $j = 2, \ldots, n$, *moreover we define* $\alpha_{n+1} = +\infty$ *and* $\alpha_{-n-1} = -\infty$, *and*

(iv.1) *the function* $f_1(z)$ *is nondecreasing in* (α_j, β_j) *if* j *is even,*

(iv.2) *the function* $f_1(z)$ *is nonincreasing in* (α_j, β_j) *if* j *is odd,*

(iv.3) *the function* $f_2(z)$ *is nondecreasing in* $(\alpha_{-j}, \beta_{-j})$ *if* j *is odd,*

(iv.4) *the function* $f_2(z)$ *is nonincreasing in* $(\alpha_{-j}, \beta_{-j})$ *if* j *is even.*

Then the following statements hold.

(a) *System* (3) *has at least* n *limit cycles if the conditions* (i)–(iii) *are satisfied.*
(b) *System* (3) *has exactly* n *limit cycles if the conditions* (i)–(iv) *are satisfied.*

Theorem 3 is proved in Section 3.

2. Proof of Theorem 1

We shall need a preliminary lemma for proving Theorem 1. This lemma is due to Xiudong Chen [3]. First we introduce some definitions and notation.

Consider the differential equation

$$\frac{dz}{dy} = F(z) - y. \tag{6}$$

The function $y = F(x)$ is called the *characteristic function*. The trajectory of (4) passing through the point $(z_0, F(z_0))$ is called the z_0 *characteristic trajectory* of Eq. (4).

If $F(z) \in \mathcal{C}^1(0, +\infty)$ and $F(0) = 0$, then the characteristic trajectory of Eq. (4) must intersect the y-axis at points $A = (0, y_A)$ and $B = (0, y_B)$ such that either $y_A < 0$ and $y_B \geqslant 0$, or $y_A \leqslant 0$ and $y_B > 0$. In the first situation the point B is called *upper* z_0 *characteristic point*, and the point A the *lower* z_0 *characteristic point*. In the second situations the points A and B are interchanged.

Lemma 4. *Let* $z_0 > 0$. *If* $y_0 \geqslant F(z_0)$ *and* $B = (0, y_B)$ *is the upper* z_0 *characteristic point of the trajectory of* (6) *passing through the point* (z_0, y_0). *If* $M = \max_{z \in [0, z_0]} F(z)$ *and* $m = \min_{z \in [0, z_0]} F(z)$, *then*

(a) $\max\{M, m + \sqrt{(m - y_0)^2 + 2z_0}\} \leqslant y_B \leqslant M + \sqrt{2z_0}$ *if* $m \leqslant y_0 \leqslant M$;
(b) $m + \sqrt{(m - y_0)^2 + 2z_0} \leqslant y_B \leqslant M + \sqrt{(M - y_0)^2 + 2z_0}$ *if* $y_0 > M$.

If $y_0 \leqslant F(z_0)$ *and* $A = (0, y_A)$ *is the lower* z_0 *characteristic point of the trajectory of* (6) *passing through the point* (z_0, y_0), *then*

(c) $m - \sqrt{2z_0} \leqslant y_A \leqslant \min\{m, M - \sqrt{(M - y_0)^2 + 2z_0}\}$ *if* $y_0 \geqslant m$;
(d) $m - \sqrt{(m - y_0)^2 + 2z_0} \leqslant y_A \leqslant M - \sqrt{(M - y_0)^2 + 2z_0}$ *if* $y_0 < m$.

Proof. Assume that $y_0 \geqslant F(z_0)$. We claim that the trajectory of (6) through the point (z_0, y_0) will stay after passing by this point over the curve $y = F(z)$. Now we prove the claim. We note that the differential system

$$\dot{z} = F(z) - y, \qquad \dot{y} = 1, \tag{7}$$

is equivalent to system (6). Since on the curve $y = F(z)$ the vector field associated to (7) is $(0, 1)$, it follows that the trajectory in the (z, y)-plane through the point (z_0, y_0) located over the curve $y = F(z)$ must remain after passing for this point over that curve. Hence the claim is proved.

Now we introduce the two *comparison differential equations*

$$\frac{dz}{dy} = M - y, \tag{8}$$

and

$$\frac{dz}{dy} = m - y. \tag{9}$$

The solution of Eq. (8) through the point (z_0, y_0) is

$$z(y) = -\frac{1}{2}(y - M)^2 + z_0 + \frac{1}{2}(y_0 - M)^2.$$

This solution intersects the straight line $z = 0$ into two points $(0, y_B^M)$ and $(0, y_A^M)$ with

$$y_B^M = M + \sqrt{2z_0 + (y_0 - M)^2} \quad \text{and} \quad y_A^M = M - \sqrt{2z_0 + (y_0 - M)^2}.$$

The solution of Eq. (9) through the point (z_0, y_0) is

$$z(y) = -\frac{1}{2}(y - m)^2 + z_0 + \frac{1}{2}(y_0 - m)^2.$$

This solution intersects the straight line $z = 0$ into two points $(0, y_B^m)$ and $(0, y_A^m)$ with

$$y_B^m = m + \sqrt{2z_0 + (y_0 - m)^2} \quad \text{and} \quad y_A^m = m - \sqrt{2z_0 + (y_0 - m)^2}.$$

Case $z_0 > 0$, $y_0 \geqslant F(z_0)$ and $m \leqslant y_0 \leqslant M$. By the comparison differential equation (8) we obtain that

$$y_B \leqslant \min_{y_0 \in [F(z_0), M]} y_B^M \leqslant M + \sqrt{2z_0}.$$

On the other hand again from the comparison differential equation (9) it follows that $y_B^m \leqslant y_B$, and since from the claim we get that $y_B \geqslant M$. Therefore $y_B \geqslant \max\{M, y_B^m\}$. This completes the proof of statement (a).

Case $z_0 > 0$, $y_0 \geqslant F(z_0)$ and $y_0 > M$. Clearly by the comparison differential equation (8) we have that $y_B \leqslant y_B^M$, and by the comparison differential equation (9) we obtain $y_B \leqslant y_B^m$. Since

$y_0 > M$ using system (7) we get that $M > y_B^m$, therefore the maximum which appears in the inequality of statement (a) has no meaning in this case. In short, statement (b) is proved.

Assume that $y_0 \leqslant F(z_0)$. We claim that the trajectory of (6) through the point (z_0, y_0) will stay after passing for this point in backward time below the curve $y = F(z)$. The proof of this claim is similar to the proof of the previous claim.

Case $z_0 > 0$, $y_0 \leqslant F(z_0)$ and $y_0 \geqslant m$. By the comparison differential equation (8) we obtain that $y_A \leqslant y_A^M$. Since from the claim we get that $y_A \leqslant m$, it follows that $y_A \leqslant \min\{m, y_A^M\}$. On the other hand from the comparison differential equation (9) we get that $y_A^m \leqslant y_A$, and since $y_0 \geqslant m$ we obtain

$$\max_{y_0 \geqslant m} y_A^m = m - \sqrt{2z_0} \leqslant y_A.$$

This completes the proof of statement (c).

Case $z_0 > 0$, $y_0 \leqslant F(z_0)$ and $y_0 < m$. Clearly by the comparison differential equation (8) we have that $y_A \leqslant y_A^M$, and by the comparison differential equation (9) we obtain $y_A^m \leqslant y_A$. Since $y_0 < m$ using system (9) we get that $m > y_A^M$, therefore the minimum which appears in the inequality of statement (c) has no meaning in this case. In short, (d) is proved. $\square$

Proof of Theorem 1. Let $y_B(z_{ij})$ $(y_A(z_{ij}))$ denote the upper (lower) characteristic point, then by statements (a) and (c) of Lemma 4 we have

$$y_B(z_{10}) < y_B(z_{20}), \quad y_A(z_{10}) < y_A(z_{20}), \quad y_B(z_{21}) < y_B(z_{11}), \quad y_A(z_{21}) < y_A(z_{11}).$$

See Fig. 1.

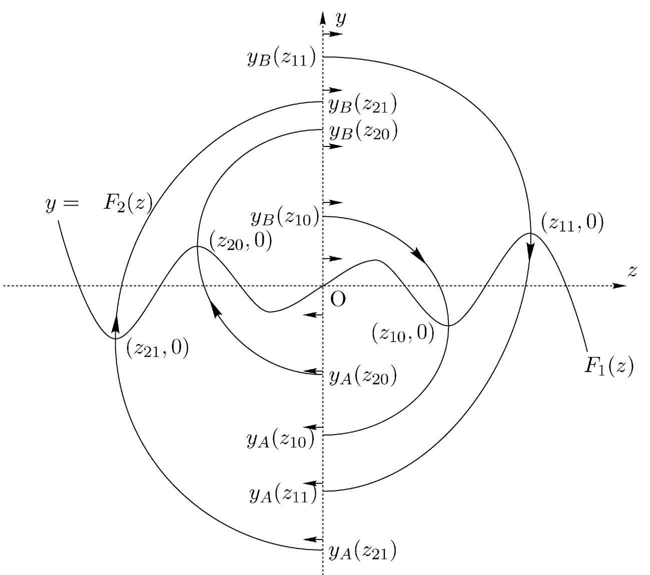

Fig. 1.

Let

$$L_1 = \overline{y_B(z_{20})y_B(z_{10})} \cup \overline{y_B(z_{10})y_A(z_{10})} \cup \overline{y_A(z_{10})y_A(z_{20})} \cup \overline{y_A(z_{20})y_B(z_{20})},$$
$$L_2 = \overline{y_B(z_{21})y_B(z_{11})} \cup \overline{y_B(z_{11})y_A(z_{11})} \cup \overline{y_A(z_{11})y_A(z_{21})} \cup \overline{y_A(z_{21})y_B(z_{21})}.$$

We note that L_1 and L_2 are the inner and the outer bound of an annular region, the vector fields can only go out of it, so by the Poincaré Annular Theorem (see [5]) there exists at least one limit cycle. □

3. Proof of Theorem 3

Let

$$L_k = \overline{y_B(\alpha_{-k})y_B(\alpha_k)} \cup \overline{y_B(\alpha_k)y_A(\alpha_k)} \cup \overline{y_A(\alpha_k)y_A(\alpha_{-k})} \cup \overline{y_A(\alpha_{-k})y_B(\alpha_{-k})}.$$

By Theorem 1, from conditions (i)–(iii) with respect to L_k the vector field associated to system (3) will point from inside (or outside) to outside (or inside) if k is odd (or even). Therefore by the Poincaré Annular Theorem there is at least one limit cycle between L_k and L_{-k}. As an example we discuss the properties of L_2 and L_3 in more detail in the following.

According with condition (iii) of this theorem and conditions (i) and (ii) of Corollary 2, if we denote by $z_{10} = \alpha_1$, $z_{11} = \alpha_2$, $z_{20} = \alpha_{-1}$, $z_{21} = \alpha_{-2}$, $M_{10} = F_1(\alpha_1)$, $m_{10} = F_1(\alpha_2)$, $M_{20} = F_2(\alpha_{-2})$, $m_{20} = F_2(\alpha_{-1})$, we have that

$$M_{20} - M_{10} \geqslant \sqrt{2\alpha_2} = \sqrt{2z_{11}} \geqslant \sqrt{2z_{10}},$$
$$m_{20} - m_{10} \geqslant \sqrt{2\alpha_{-2}} = \sqrt{2z_{21}} \geqslant \sqrt{2z_{20}}.$$

Therefore

$$M_{10} + \sqrt{2z_{10}} \leqslant M_{20}, \quad \text{and} \quad m_{10} + \sqrt{2z_{20}} \leqslant m_{20}.$$

This is the condition (i) of Corollary 2.

Let $z_{12} = \alpha_3$, $z_{22} = \alpha_{-3}$, $\alpha_3 > \alpha_2 > 0$, $\alpha_{-3} > \alpha_{-2} > 0$, $M_{11} = F_1(\alpha_3)$, $m_{21} = F_2(\alpha_{-3})$. Then condition (iii) implies

$$M_{11} - M_{20} \geqslant \sqrt{2z_{22}} \geqslant \sqrt{2z_{21}},$$
$$m_{10} - m_{21} \geqslant \sqrt{2z_{12}} \geqslant \sqrt{2z_{11}}.$$

Since $M_{21} = \max_{z\in[0,z_{21}]} F_2(z) = M_{20}$ and $m_{11} = \min_{z\in[0,z_{11}]} F_1(z) = m_{10}$, we have that

$$M_{11} - M_{21} \geqslant \sqrt{2z_{21}} \quad \text{and} \quad m_{11} - m_{21} \geqslant \sqrt{2z_{11}}.$$

This is the condition (ii) of Corollary 2.

In short L_2 and L_3 are the inner and outer boundaries of an annular region; so between L_2 and L_3 there is at least one limit cycle.

In a similar way can be proved that in the annular region limited by L_{k-1} and L_k with $2 \leqslant k \leqslant n$, there is at least one limit cycle.

Since the origin is a stable singular point in the topological disc limited by L_2 there exists at least one limit cycle. Moreover, since the infinity is an attractor (or a repeller) if n is odd (or even), so outside the region limited by L_n there exists at least one limit cycle.

In short we have proved statement (a) of Theorem 3.

For proving statement (b) of Theorem 3 we need some preliminary results.

Lemma 5. *Let $L = \{(x(t), y(t))\colon t \in \mathbb{R}\}$ be a limit cycle of system* (3) *of period T, let* $\operatorname{div}(x, y)$ *be its divergence, and $\gamma = \int_0^T \operatorname{div}(x(t), y(t))\,dt$. If $\gamma < 0$ then L is stable, and if $\gamma > 0$ then L is unstable.*

Lemma 5 is well known, for a proof see for instance [5].

Lemma 6. *If for system* (3) *with divergence* $\operatorname{div}(x, y)$ *there exist $0 \leqslant a < \zeta < b$ such that*

(i) *$F(a) = F(b)$,*
(ii) *$F'(x) = f(x) > 0$ (respectively < 0) if $x \in (a, \zeta)$, and $F'(x) = f(x) < 0$ (respectively > 0) if $x \in (\zeta, b)$,*

then $\int_0^s \operatorname{div}(x(t), y(t))\,dt = -\int_a^b f(x)\,dx > 0$ (respectively < 0), where $\{(x(t), y(t))\colon 0 \leqslant t \leqslant s\}$ is a solution curve of system (3) *such that its projection on the x-axis is the interval $[a, b]$.*

Proof. See Lemma 1 of [19]. □

Lemma 7. *Assume that the conditions of Lemma* 6 *outside* (*respectively inside*) *the parentheses hold.*

(a) *If we have two solution curves $S_i = \{(x_i(t), y_i(t))\}$ for $i = 1, 2$, located over the characteristic curve $y = F(x)$ with $a \leqslant x_i(t) \leqslant b$, then*

$$\int_{S_2} \operatorname{div}\bigl(x_2(t), y_2(t)\bigr)\,dt < \int_{S_1} \operatorname{div}\bigl(x_1(t), y_1(t)\bigr)\,dt \tag{10}$$

if S_2 is over (*respectively below*) *S_1.*

(b) *If we have two solution curves $S_i = \{(x_i(t), y_i(t))\}$ for $i = 1, 2$, located below the characteristic curve $y = F(x)$ with $a \leqslant x_i(t) \leqslant b$, then* (10) *is satisfied if S_2 is below* (*respectively over*) *S_1.*

Proof. See Lemma 3 of [19]. □

Lemma 8. *Assume that there exists a subinterval $[\alpha, \beta]$ in the half positive x-axis for which the system* (3) *is defined in the strip $\{(x, y)\colon x \in [\alpha, \beta],\ y \in \mathbb{R}\}$, and that for $x \in (\alpha, \beta)$ we have*

(i) *$f(x)$ is nondecreasing* (*respectively nonincreasing*);
(ii) *$f(x) \geqslant 0$ (respectively $\leqslant 0$).*

Then along the two solution curves $S_i = \{(x_i(t), y_i(t))\}$ passing through the points $(\beta_i^*, 0)$, for $i = 1, 2$, with $\alpha < \beta_1^* < \beta_2^* \leqslant \beta$, and intersecting twice the straight line $x = \alpha$ we have that

$$\int\limits_{S_2} \operatorname{div}\bigl(x_2(t), y_2(t)\bigr)\, dt \leqslant \quad (\textit{respectively} \ \geqslant) \quad \int\limits_{S_1} \operatorname{div}\bigl(x_1(t), y_1(t)\bigr)\, dt.$$

Proof. See Lemma 4 of [19]. □

Preparation Theorem. *If for system* (3) *the conditions* (i)–(iv) *are satisfied, and $f(0) > 0$, then the following statements hold.*

(a) *Suppose that there exist two limit cycles l and l' with l contained in the bounded region limited by l' (we shall write simply $l \subset l'$), contained into the region $\beta_{-2k} \leqslant x \leqslant \beta_{2k}$ (respectively $\beta_{-2k-1} \leqslant x \leqslant \beta_{2k+1}$), and intersecting the straight lines $x = \alpha_{2k}$ and $x = \alpha_{-2k}$ (respectively $x = \alpha_{2k+1}$ and $x = \alpha_{-2k-1}$). Then*

$$\int\limits_{l} \operatorname{div}(x, y)\, dt > \quad (\textit{respectively} \ <) \quad \int\limits_{l'} \operatorname{div}(x, y)\, dt.$$

(b) *Suppose that there exists a limit cycle l, contained into the region $\alpha_{-2k-1} \leqslant x \leqslant \beta_{2k+1}$ (respectively $\beta_{-2(k+1)} \leqslant x \leqslant \beta_{2(k+1)}$), and intersecting the straight lines $x = \beta_{2k}$ and $x = \beta_{-2k}$ (respectively $x = \beta_{2k+1}$ and $x = \beta_{-2k-1}$), where $k \geqslant 1$. Then*

$$\int\limits_{l} \operatorname{div}(x, y)\, dt < 0 \quad (\textit{respectively} \ > 0).$$

Proof. According to Ref. [20], first we decompose the limit cycles l and l' into ordered arcs as follows (see Fig. 2):

$$l(x \geqslant 0) = \left[\bigcup_{j=1}^{k} (S_j \cup l_j)\right] \cup r_+, \qquad l(x \leqslant 0) = \left[\bigcup_{j=-1}^{-k} (S_j \cup l_j)\right] \cup r_-,$$

$$l'(x \geqslant 0) = \left[\bigcup_{j=1}^{k} \bigl(S'_j \cup l'_j\bigr)\right] \cup r'_+, \qquad l'(x \leqslant 0) = \left[\bigcup_{j=-1}^{-k} \bigl(S'_j \cup l'_j\bigr)\right] \cup r'_-;$$

or as follows (see Fig. 3):

$$l(x \geqslant 0) = \left[\bigcup_{j=1}^{2k-1} (l_j \cup S_j)\right] \cup l_{2k} \cup r_+, \qquad l(x \leqslant 0) = \left[\bigcup_{j=-1}^{-2k+1} (l_j \cup S_j)\right] \cup l_{-2k} \cup r_-,$$

$$l'(x \geqslant 0) = \left[\bigcup_{j=1}^{2k-1} \bigl(l'_j \cup S'_j\bigr)\right] \cup l'_{2k} \cup r'_+, \qquad l'(x \leqslant 0) = \left[\bigcup_{j=-1}^{-2k+1} \bigl(l'_j \cup S'_j\bigr)\right] \cup l'_{-2k} \cup r'_-.$$

In the case outside the parentheses, from Lemmas 5–7, Fig. 2 and the corresponding ordered decomposition of l and l' we get statement (a).

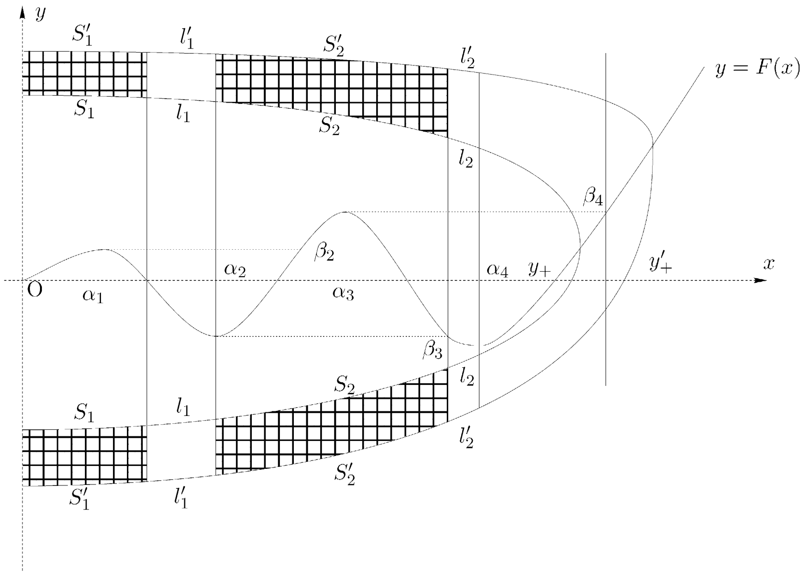

Fig. 2.

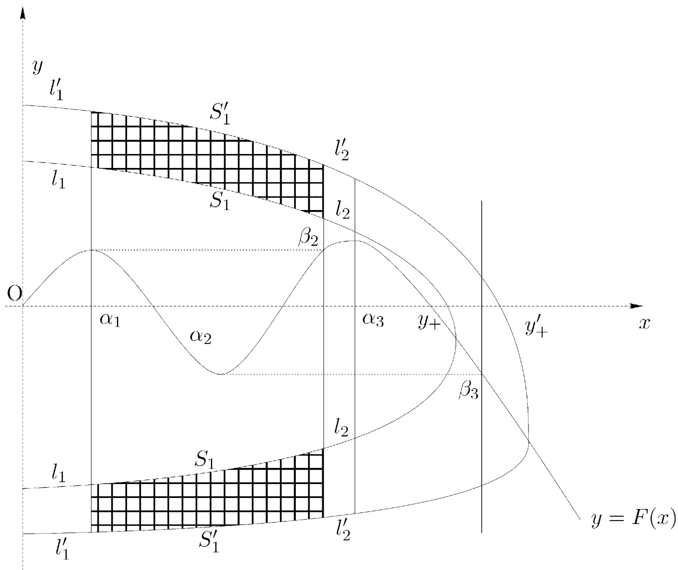

Fig. 3.

In the case inside the parentheses, from Lemmas 5–7, Fig. 3 and the corresponding ordered decomposition of l and l' we get statement (a).

In the case outside the parentheses, from Lemma 5, Fig. 2 and the corresponding ordered decomposition of l, we get statement (b).

In the case inside the parentheses, from Lemma 5, Fig. 3 and the corresponding ordered decomposition of l, we get statement (b). □

Now using the Preparation Theorem we prove statement (b) of Theorem 3.

We must prove that between L_k and L_{k-1} (for $k = 3, \dots, n$) there exists exactly one limit cycle, which is unstable (respectively stable), if k is even (respectively odd).

By Lemma 5, in the interior of L_2 there exists a limit cycle l_1 which is unstable, i.e.

$$\int_{l_1} \operatorname{div}(x, y)\, dt > 0.$$

By the statement (a) of the Preparation Theorem, if there exist two limit cycles $l_2 \subset l_3$ in the region $\beta_{-2} < x < \beta_2$, intersecting the straight lines $x = \alpha_2$ and $x = \alpha_{-2}$, since $l_1 \subset l_2$, then we have

$$0 \geqslant \int_{l_2} \operatorname{div}(x, y)\, dt > \int_{l_3} \operatorname{div}(x, y)\, dt.$$

So l_3 is stable. By perturbation technique it can be proved that l_2 is also stable. Therefore there exists at most one limit cycle in the region $\beta_{-2} < x < \beta_2$, intersecting the straight lines $x = \alpha_2$ and $x = \alpha_{-2}$, which is stable.

By the statement (b) of the Preparation Theorem, if there is no limit cycle in the region $\beta_{-2} < x < \beta_2$ intersecting the straight lines $x = \alpha_2$ and $x = \alpha_{-2}$, then there exists a limit cycle in the region $\alpha_{-3} < x < \alpha_3$ intersecting the straight lines $x = \beta_2$ and $x = \beta_{-2}$, which is stable. Therefore between L_2 and L_3 there exists exactly one limit cycle, which is stable.

In a similar way, by the Preparation Theorem, if there exist two limit cycles $l_3 \subset l_4$ in the region $\beta_{-3} < x < \beta_3$ intersecting the straight lines $x = \alpha_3$ and $x = \alpha_{-3}$, then

$$0 \leqslant \int_{l_3} \operatorname{div}(x, y)\, dt < \int_{l_4} \operatorname{div}(x, y)\, dt.$$

So l_4 is unstable. By perturbation technique it can be proved that l_3 is also unstable. Therefore there exists at most one limit cycle in the region $\beta_{-3} < x < \beta_3$ intersecting the straight lines $x = \alpha_3$ and $x = \alpha_{-3}$, which is unstable.

If there is no limit cycle in the region $\beta_{-3} < x < \beta_3$ intersecting the straight lines $x = \alpha_3$ and $x = \alpha_{-3}$, then there exists a limit cycle in the region $\alpha_{-4} < x < \alpha_4$ intersecting the straight lines $x = \beta_3$ and $x = \beta_{-3}$, then by the statement (b) of the Preparation Theorem, it is unstable. Therefore there exists exactly one limit cycle in the region $\alpha_{-4} < x < \alpha_4$ intersecting the straight lines $x = \beta_3$ and $x = \beta_{-3}$, which is unstable. In short, the conclusion is that at exact one limit cycle between L_3 and L_4, which is unstable.

Similarly it can be proved, that between L_k and L_{k-1} (for $k = 3, \dots, n$) there are exactly one limit cycle, which is unstable (respectively stable), if k is even (respectively odd).

By [17] in the interior of L_2 there exists exactly one limit cycle, which is unstable.

When n is even (or odd), the infinity is a repeller (or an attractor), so outside L_n there exists at least one limit cycle. Since $F(x)$ is monotone when $x \geqslant \alpha_n$ and $x \leqslant \alpha_{-n}$, by Lemma 7 outside L_n there exists exactly one limit cycle, which is stable (respectively unstable).

In short statement (b) of Theorem 3 is proved.

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